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Geometric and Semigeometric Approximation of Spectral Projections

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1. INTRODUCTION

Let T be a closed operator in a Banach space, and let a closed Jordan curve Γ isolate a finite number of eigenvalues of T (of finite algebraic multiplicities) from the rest of the spectrum of T. In order to find the eigenvectors of T corresponding to eigenvalues inside Γ, it is useful to consider the spectral projection P associated with the spectrum of T inside Γ. If T is approximated by a sequence of closed operators T_n in a certain sense, then for large enough n, there are only finitely many eigenvalues of each T_n inside Γ. Let P_n be the spectral projection of T_n associated with these eigenvalues so that P_n is close enough to P. Let T_n have only one eigenvalue λ_n inside Γ. This is always the case if Γ encloses only one simple eigenvalue of T. Then we can use an infinite series expansion for P in terms of P_n and the reduced resolvent S_n (w.r.t. T_n and λ_n), given by Kato in [5]. By truncating the infinite series for P at various stages, we get approximations for P. In case Γ encloses only a simple eigenvalue λ of T, an eigenvector of T can be obtained by evaluating the series for P at any vector. If, in particular, it is evaluated at an eigenvector ϕ_n of T_n associated with the simple eigenvalue λ_n, then the eigenvector $P\phi_n$ is the same as the one obtained by evaluating the series for PP_n at ϕ_n, namely, $PP_n\phi_n$. Moreover, it is more convenient and economical to consider the series for PP_n obtained from the series for P.

In Section 2, we set up the notation. In Section 3, we consider T as a perturbation of an operator T_0 by $T - T_0$, and give some bounds for the jth approximation of PP_0, $j = 0, 1, 2, \ldots$. This work generalizes some of the previous results in [6] for $j = 0$, 1, and 2. In Section 4, we prove that under

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the assumption of norm convergence, every approximation of PP_n improves upon the earlier one, whereas under the assumption of collectively compact convergence this improvement is only in steps of 2. In the last section, we consider approximations of $P\phi_n$. They are compared with the approximations of another eigenvector ϕ given by the Rayleigh–Schrodinger series [4, 7]. Under norm convergence, both the eigenvectors $P\phi_n$ and ϕ are approximated in a geometric fashion. On the other hand, if T_n converges to T in a collectively compact manner, then $P\phi_n$ and ϕ are both approximated in a semigeometric fashion [7, 8]. This is borne out in the numerical experiments involving two integral operators with kernels

$$\begin{array}{ll} s(1-t), & \text{if } \; 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & \text{if } \; 0 \leqslant t < s \leqslant 1 \end{array}$$

and

$$e^{st}, \qquad 0 \leqslant s, t \leqslant 1.$$

It is observed that in the case of $P\phi_n$ the zeroth and the first approximations are of the same order, while the second approximation improves upon them. On the other hand, in the case of the Rayleigh–Schrodinger eigenvector ϕ, the first approximation is an improvement on the zeroth approximation but the second approximation is of the same order as the first.

2. Preliminaries

Let X be a complex Banach space and $L(X)$ the algebra of all bounded (linear) operators on X. Let T_0 be a closed operator with domain $D_0 \subset X$ and λ_0 be an isolated eigenvalue of T_0, of finite algebraic multiplicity and index 1. Let Γ be a closed Jordan curve enclosing λ_0 but not enclosing any other part of the spectrum $\sigma(T_0)$ of T_0.

For z belonging to the resolvent set $\rho(T_0)$ of T_0, let $R_0(z) = (T_0 - zI)^{-1}$ be the resolvent of T_0. Then the spectral projection P_0 and the reduced resolvent S_0 associated with T_0 and λ_0 are defined as follows:

$$P_0 = -\frac{1}{2\pi i} \int_\Gamma R_0(z)\, dz, \qquad S_0 = \lim_{z \to \lambda_0} R_0(z)(I - P_0).$$

Also, $R_0(z)$ has the Laurent series expansion

$$R_0(z) = \sum_{k=-1}^{\infty} (z - \lambda_0)^k S_0^{k+1},$$

where $S_0^0 = -P_0$, so that λ_0 is a pole of $R_0(z)$ of order 1.

Let T be a closed operator with domain D containing D_0 and such that

$$\max_{z\in\Gamma} r_\sigma((T_0-T)\,R_0(z))<1,$$

where $r_\sigma((T_0-T)\,R_0(z))$ denotes the spectral radius of the bounded operator $(T_0-T)\,R_0(z)$. Then $\Gamma\subset\rho(T)$ and $R(z)=(T-zI)^{-1}$ can be represented by the second Neumann series

$$R(z)=R_0(z)\sum_{k=0}^{\infty}[(T_0-T)\,R_0(z)]^k,\qquad z\in\Gamma.$$

Consider the spectral projection

$$P=-\frac{1}{2\pi i}\int_\Gamma R(z)\,dz$$

associated with the spectrum of T inside Γ. The number of eigenvalues of T (counted according to algebraic multiplicities) inside Γ is equal to the algebraic multiplicity of λ_0.

Let $k\geqslant 0$ be an integer and let $(*)$ denote the conditions

$$p_1+\cdots+p_{k+1}=k,$$
$$p_j\geqslant 0,\qquad j=1,\dots,k+1,$$

where $p_1,\dots,p_{k+1}$ are integers.

If, in addition, exactly q of the integers $p_1,\dots,p_{k+1}$ are zero, we denote the above conditions by $(*,q)$.

The number n_{k+1} of the ordered $(k+1)$-tuples $(p_1,\dots,p_{k+1})$ satisfying $(*)$ is the coefficient of x^k in the binomial expansion of $(1-x)^{-(k+1)}$. Thus,

$$n_{k+1}=(2k)!/k!\,k!.$$

It can be immediately seen that the series

$$\sum_{k=0}^{\infty} n_{k+1}x^k \tag{2.1}$$

converges for $|x|<\frac{1}{4}$.

Using the Neumann series of $R(z)$, we have

$$P-P_0=-\sum_{k=1}^{\infty}\sum_{(*)}S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}. \tag{2.2}$$

Multiplying both sides of (2.2) by P_0, we have

$$PP_0 - P_0 = -\sum_{k=1}^{\infty} \sum_{(*)} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} P_0$$

$$= -\sum_{k=1}^{\infty} \sum_{(*),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} \qquad (2.3)$$

since $S_0 P_0 = 0$ and $S_0^0 = -P_0$. If ϕ_0 is an eigenvector of T_0 corresponding to the eigenvalue λ_0, then

$$(P - P_0)(\phi_0) = (PP_0 - P_0)(\phi_0),$$

but the series for $(PP_0 - P_0)$ in (2.3) has less number of terms than the one for $(P - P_0)$ in (2.2). We shall, therefore, consider the series in (2.3). This will facilitate the approximate calculation of $P\phi_0$ and would give better error bounds.

We write

$$P_0^{(0)} = P_0,$$

$$P_0^{(k)} = -\sum_{(*),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}, \qquad k \geqslant 1,$$

so that

$$PP_0 = P_0 + \sum_{k=1}^{\infty} P_0^{(k)}.$$

Let the jth approximation of PP_0 be denoted by

$$P_0^j = \sum_{k=0}^{j} P_0^{(k)}, \qquad j \geqslant 0.$$

We shall give estimates for $\|PP_0 - P_0^j\|$ in terms of the following quantities:

$$\varepsilon_0 = \|(T_0 - T)\, P_0\|,$$

$$\alpha_0 = \|(T_0 - T)\, S_0\|,$$

$$\gamma_0 = \max\{\|S_0\|\, \varepsilon_0, \alpha_0\},$$

$$u_0 = \|S_0\|\, \varepsilon_0 \gamma_0,$$

$$\beta_{0,k} = \frac{\|(T_0 - T)\, S_0^k (T_0 - T)\, S_0\|}{\|S_0\|^{k-1}}, \qquad k = 1, 2, \ldots,$$

$$\delta_{0,k} = \max\{\|S_0\|\, \varepsilon_0 \alpha_0, \beta_{0,1}, \ldots, \beta_{0,k}\}, \qquad k = 1, 2, \ldots,$$

and

$$\begin{aligned} v_0 &= \max\{(\gamma_0^2 u_0)^{1/4}, \sqrt{\beta_{0,1}}\} \\ &= \max\{\|S_0\|\,\varepsilon_0, \sqrt{\|S_0\|\ \varepsilon_0\alpha_0}, (\|S_0\|^3\,\varepsilon_0^3\alpha_0)^{1/4}, (\|S_0\|\,\varepsilon_0\alpha_0^3)^{1/4}, \sqrt{\beta_{0,1}}\} \end{aligned}$$

We shall now explain the motivation behind introducing the above quantities. Let λ be a simple eigenvalue of T isolated by a closed Jordan curve Γ. Let T_n be a sequence in $L(X)$ converging to T in the norm ($\|T_n - T\| \to 0$) or in the collectively compact fashion ($T_n \to^{\text{c.c.}} T$). Then, for all large enough n, the curve Γ is contained in the resolvent set of T_n, and there is only a simple eigenvalue λ_n of T_n inside Γ. Let $R_n(z) = (T_n - zI)^{-1}$ for $z \in \rho(T_n)$. We have

$$r_\sigma((T_n - T)\,R_n(z)) \to 0$$

uniformly for $z \in \Gamma$. Then, for large enough n, we have

$$\max_{z \in \Gamma} r_\sigma((T_n - T)\,R_n(z)) < 1.$$

We can, therefore, replace T_0 by T_n, P_0 by P_n, and S_0 by S_n in the earlier setting.

In case $\|T_n - T\| \to 0$, we see that $\|P_n\|$ and $\|S_n\|$ are bounded (Chatelin [2, Lemma 5.19]) and hence

$$\varepsilon_n = \|(T_n - T)\,P_n\| \to 0 \qquad \text{and} \qquad \alpha_n = \|(T_n - T)\,S_n\| \to 0.$$

If $T_n \to^{\text{c.c.}} T$, then $P_n \to^{\text{c.c.}} P$, $(T_n - T)\,S_n \to^{\text{c.c.}} 0$ (Anselone [1, Theorem 4.16]). Hence $\varepsilon_n \to 0$, while α_n is only bounded (Anselone [1, Proposition 4.2]). In this case, α_n may not tend to zero, but the following proposition shows that for any fixed $k \geqslant 1$,

$$\beta_{n,k} = \frac{\|(T_n - T)\,S_n^k(T_n - T)\,S_n\|}{\|S_n\|^{k-1}} \to 0.$$

Note that $(1/\|S_n\|) \leqslant \text{dist}(\lambda_n, \sigma(T_n) - \{\lambda_n\}) \leqslant 2\,\|T_n\|$, which is bounded.

PROPOSITION 2.1. *Let $T_n \to^{\text{c.c.}} T$ and $k \geqslant 1$. Then, for $z \in \rho(T)$, we have*

$$\|(T_n - T)\,R_n(z)^k\,(T_n - T)\,R_n(z)\| \to 0,$$

and hence

$$\|(T_n - T)\,S_n^k(T_n - T)\,S_n \to 0.$$

Proof. $T_n \to^{\text{c.c.}} T$ implies that the set

$$\{(T_n - T)\, R(z),\ z \in \Gamma,\ n \in \mathbb{N}\}$$

is collectively compact and

$$(T_n - T)\, R(z)^k \xrightarrow{\text{c.c.}} 0 \qquad \text{for} \quad z \in \Gamma$$

(Anselone [1, Proposition 4.2(2)]). Hence

$$\|(T_n - T)\, R(z)^k (T_n - T)\, R(z)\| \to 0 \qquad \text{for} \quad z \in \Gamma$$

(Anselone [1, Proposition 1.7]). From the second Neumann series of $R_n(z)$, we have

$$\begin{aligned} R_n(z) - R(z) = {} & R(z)(T - T_n)\, R(z) \\ & + R(z)(1 + (T - T_n)\, R(z)) \sum_{j=1}^{\infty} [(T - T_n)\, R(z)]^{2j}. \end{aligned}$$

Therefore,

$$\|(T_n - T)\, R(z)^{j-1}\, (R_n(z) - R(z))\| \to 0 \qquad \text{for} \quad z \in \Gamma \quad \text{and} \quad 1 \leqslant j \leqslant k.$$

Now,

$$\begin{aligned} R_n(z)^k &= (R_n(z) - R(z) + R(z))^k \\ &= (R_n(z) - R(z))\, R_n(z)^{k-1} + R(z)(R_n(z) - R(z))\, R_n(z)^{k-2} \\ &\quad + R(z)^2\, (R_n(z) - R(z))\, R_n(z)^{k-3} + \\ &\quad\ \vdots \\ &\quad + R(z)^{k-2}\, (R_n(z) - R(z))\, R_n(z) \\ &\quad + R(z)^{k-1}\, (R_n(z) - R(z)) + R(z)^k. \end{aligned}$$

Hence it follows that

$$\|(T_n - T)\, R_n(z)^k\, (T_n - T)\, R_n(z)\| \to 0 \qquad \text{for} \quad z \in \Gamma.$$

Since $S_n = \lim_{z \to \lambda_n} R_n(z)(I - P_n)$, we see that

$$\begin{aligned} & \|(T_n - T)\, S_n^k (T_n - T)\, S_n\| \\ & \qquad \leqslant \max_{z \in \Gamma} \|(T_n - T)\, R_n(z)^k\, (I - P_n)(T_n - T)\, R_n(z)(I - P_n)\| \to 0. \end{aligned}$$

This completes the proof. ∎

3. Estimates for $\|PP_0 - P_0^j\|$

In this section, we give bounds for the difference between PP_0 and its jth approximation P_0^j. We begin with a crucial lemma whose proof is combinatorial in character.

Lemma 3.1. *Let $k+1 > q \geqslant i \geqslant 2$ be integers. Let $p_1,\dots, p_{k+1}$ satisfy $(*, q)$. Then*

$$\|S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\| \leqslant \|P_0\|\, \|S_0\|^{i-1}\, \varepsilon_0^{i-1} \gamma_0^{k-i+1}. \tag{3.1}$$

In particular, if $k+1 > 4i-3$ and $p_{k+1} = 0$, then

$$\|S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\| \leqslant \|P_0\|\, \|S_0\|^{i-1}\, \varepsilon_0^{i-1} \gamma_0^{3i-3} v_0^{k-4i+4}. \tag{3.2}$$

Also, if $k+1 > 2q-1$ and $p_{k+1} = 0$, then

$$\|S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\|$$
$$\leqslant \begin{cases} \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{q-2} (\delta_{0,q})^{(k-2q+3)/2}, & k+1 \text{ even} \quad (3.3)\\ \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{q-1} (\delta_{0,q})^{(k-2q+2)/2}, & k+1 \text{ odd}. \quad (3.4)\end{cases}$$

Proof. If $p_j = 0$, then

$$\|(T_0 - T)\, S_0^{p_j}\| = \|(T_0 - T)\, P_0\| = \varepsilon_0,$$

while if $p_j > 0$, then

$$\|(T_0 - T)\, S_0^{p_j}\| \leqslant \|(T_0 - T)\, S_0\|\, \|S_0\|^{p_j - 1} = \|S_0\|^{p_j-1}\, \alpha_0.$$

Also, if $p_j = p_{j+1} = 1$, then

$$\|(T_0 - T)\, S_0^{p_j}(T_0 - T)\, S_0^{p_{j+1}}\| = \|[(T_0 - T)\, S_0]^2\| = \beta_{0,1}.$$

Proof of (3.1). *Case* 1: $p_1 > 0$. Since q p_j's are zero and $(k+1-q)$ p_j's are greater than zero and their sum is equal to k, it follows that

$$\begin{aligned}\|S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\| &\leqslant \varepsilon_0^q \alpha_0^{k-q}\, \|S_0\|^{\Sigma p_j - (k-q)}\\ &= \|S_0\|^q\, \varepsilon_0^q \alpha_0^{k-q}\\ &\leqslant \|P_0\|\, \|S_0\|^{i-1}\, \varepsilon_0^{i-1} \gamma_0^{k-i+1},\end{aligned} \tag{3.5}$$

since $1 \leqslant \|P_0\|$, $\|S_0\|\, \varepsilon_0 \leqslant \gamma_0$ and $\alpha_0 \leqslant \gamma_0$.

Case 2: $p_1 = 0$. Among $p_2, p_3,\dots, p_{k+1}$, $(q-1)$ p_j's are zero and $(k-q+1)$ are nonzero. Hence

$$\begin{aligned}\|S_0^{p_1}(T_0-T)S_0^{p_2}\cdots(T_0-T)S_0^{p_{k+1}}\| &\leqslant \|P_0\|\,\varepsilon_0^{q-1}\alpha_0^{k-q+1}\,\|S_0\|^{\Sigma p_j-(k-q+1)}\\ &= \|P_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{k-q+1}\\ &\leqslant \|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\gamma_0^{k-i+1}. \qquad (3.6)\end{aligned}$$

This proves (3.1).

For proving (3.2), assume that $k+1>4q-3$ and $p_{k+1}=0$. We shall first show that

$$\|S_0^{p_1}(T_0-T)S_0^{p_2}\cdots(T_0-T)S_0^{p_{k+1}}\|$$
$$\leqslant\begin{cases}\|S_0\|^q\,\varepsilon_0^q\alpha_0^{3q-3}\beta_{0,1}^{(k-4q+3)/2}, & \text{if } k+1 \text{ is even}, \quad p_1>0 \qquad (3.7)\\ \|S_0\|^q\,\varepsilon_0^q\alpha_0^{3q-2}\beta_{0,1}^{(k-4q+2)/2}, & \text{if } k+1 \text{ is odd}, \quad p_1>0 \qquad (3.8)\\ \|P_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{3q-4}\beta_{0,1}^{(k-4q+5)/2}, & \text{if } k+1 \text{ is even}, \quad p_1=0 \qquad (3.9)\\ \|p_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{3q-3}\beta_{0,1}^{(k-4q+4)/2}, & \text{if } k+1 \text{ is odd}, \quad p_1=0. \qquad (3.10)\end{cases}$$

Let $p_1>0$ and $(k+1)$ be even. It follows by the pigeon-hole principle that the number of p_j's greater than 1 is at most $k-(k+1-q)=q-1$. Since the number of p_j's equal to zero is q and since $p_{k+1}=0$, we see that among $p_1,\dots,p_k$, the maximum number of p_j's not equal to 1 is $(q-1)+(q-1)=2q-2$.

Consider the pairs (p_j,p_{j+1}), $j=2,4,\dots,k-1$. Among these $(k-1)/2$ pairs, at least

$$(k-1)/2-(2q-2)=(k-4q+3)/2$$

pairs must have both $p_j=p_{j+1}=1$. Each such pair contributes one $\beta_{0,1}$ and each p_j which is zero contributes one ε_0. The remaining p_j's excluding p_1 are

$$k-(k-4q+3)-q=3q-3$$

in number. Since every such p_j contributes $\alpha_0\|S_0\|^{p_j-1}$, they contribute $\alpha_0^{3q-3}\|S_0\|^{\Sigma(p_j-1)}$ all together, where the summation is over the above $(3q-3)$ terms. Adding $\|S_0\|^{p_1}$ which comes from the first term, we see that the exponent of $\|S_0\|$ becomes

$$k-(k-4q+3)-(3q-3)=q.$$

This proves (3.7).

Now, let $p_1>0$ and $(k+1)$ be odd. As in the previous case, the maximum number of p_j's among $p_1,..., p_{k+1}$ which are not equal to 1 is $(q-1)+q = 2q-1$. Also, among the $k/2$ pairs (p_j, p_{j+1}), $j=2, 4,..., k$, at least

$$k/2-(2q-1)=(k-4q+2)/2$$

pairs must have both $p_j = p_{j+1} = 1$. Since

$$k-(k-4q+2)-q=3q-2,$$

it can be seen that (3.8) holds.

For the case $p_1=0$ and $(k+1)$ even, we note that among $p_2,..., p_k$ exactly $(q-2)$ p_j's are zero and each will contribute one ε_0. The maximum number of p_j's among $p_2,..., p_k$ which are not equal to 1 is

$$(q-1)+(q-2)=2q-3.$$

Also, among the $(k-1)/2$ pairs (p_j, p_{j+1}), $j=2, 4,..., k-1$, at least

$$(k-1)/2-(2q-3)=(k-4q+5)/2$$

pairs have both $p_j = p_{j+1} = 1$. Since

$$k-(k-4q+5)-(q-1)=3q-4,$$

it can be seen again that (3.9) holds.

The case "$p_1=0$ and $k+1$" odd can be treated similarly.

Since $1 \leqslant \|P_0\|$, $\|S_0\| \varepsilon_0 \alpha_0 \leqslant \delta_{0,1}$, $\beta_{0,1} \leqslant \delta_{0,1}$ and $\sqrt{\beta_{0,1}} \leqslant \alpha_0$, it follows from (3.7)–(3.10) that if $k+1 > 4q-3$ and $p_{k+1}=0$, then

$$\begin{aligned}\|S_0^{p_1}(T_0-T)\, S_0^{p_2} \cdots (T_0-T)\, S_0^{p_{k+1}}\| \\ \leqslant \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{3q-3} \delta_{0,1}^{(k-4q+4)/2}.\end{aligned} \tag{3.11}$$

Also, from (3.5) and (3.6) it follows that in general

$$\|S_0^{p_1}(T_0-T)\, S_0^{p_2} \cdots (T_0-T)\, S_0^{p_{k+1}}\| \leqslant \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \gamma_0^{k-q+1}. \tag{3.12}$$

Coming now to the proof of (3.2), let $(k+1) > 4i-3$ and $p_{k+1}=0$.

Case 1: $k+1 \leqslant 4q-3$. By (3.12), we have

$$\begin{aligned}&\|S_0^{p_1}(T_0-T)\, S_0^{p_2} \cdots (T_0-T)\, S_0^{p_{k+1}}\| \\ &\leqslant \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \gamma_0^{k-q+1} \\ &= \|P_0\|\, \|S_0\|^{i-1}\, \varepsilon_0^{i-1} (\|S_0\|\, \varepsilon_0)^{q-i}\, \gamma_0^{k-q+1} \\ &\leqslant \|P_0\|\, \|S_0\|^{i-1}\, \varepsilon_0^{i-1} \gamma_0^{3i-3} v_0^{k-4i+4}\end{aligned}$$

since $\|S_0\|\, \varepsilon_0 \leqslant \gamma_0$, $\alpha_0 \leqslant \gamma_0$ and $(\gamma_0^2 u_0)^{1/4} \leqslant v_0$.

Case 2: $(k+1) > 4q-3$. By (3.11), we have

$$\begin{aligned}
&\|S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\| \\
&\quad\leqslant \|P_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{3q-3}\delta_{0,1}^{(k-4q+4)/2} \\
&\quad= \|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\alpha_0^{3i-3}(\alpha_0^3\,\|S_0\|\,\varepsilon_0)^{q-i}\,\delta_{0,1}^{(k-4q+4)/2} \\
&\quad\leqslant \|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\gamma_0^{3i-3}v_0^{k-4i+4}
\end{aligned}$$

since $\alpha_0 \leqslant \gamma_0$, $\delta_{0,1} \leqslant v_0^2$ and $(\alpha_0^3\,\|S_0\|\,\varepsilon_0) \leqslant v_0^4$.

Proof of (3.3) *and* (3.4). Let $k+1 > 2q-1$ and $p_{k+1}=0$. Assume first that $k+1$ is even.

Case 1: $p_1 > 0$. Consider the $(k-1)/2$ pairs (p_j, p_{j+1}), $j = 2, 4,\ldots, k-1$. Since among $p_2,\ldots, p_k$ exactly $(q-1)$ p_j's are zero, at least $(k-1)/2-(q-1)=(k-2q+1)/2$ pairs will have both p_j and p_{j+1} greater than zero. In such a case,

$$\begin{aligned}
\|(T_0-T)\,S_0^{p_j}(T_0-T)\,S_0^{p_{j+1}}\| &= \|(T_0-T)\,S_0^{p_j}(T_0-T)\,S_0S_0^{p_{j+1}-1}\| \\
&\leqslant \|S_0\|^{p_j-1}\,\beta_{0,p_j}\,\|S_0\|^{p_{j+1}-1}.
\end{aligned}$$

Since $p_1,\ldots, p_{k+1}$ satisfy $(*, q)$, it follows that $p_j \leqslant q$ for $j=1,\ldots, k+1$. Hence

$$\|(T_0-T)\,S_0^{p_j}(T_0-T)\,S_0^{p_{j+1}}\| \leqslant \|S_0\|^{p_j+p_{j+1}-2}\delta_{0,q}.$$

In the remaining $(q-1)$ pairs, $(q-1)$ p_j's are zero and $(q-1)$ p_j's are greater than zero; if $p_j=0$, then

$$\|(T_0-T)\,S_0^{p_j}\| = \|S_0\|\,\varepsilon_0\,\|S_0\|^{p_j-1}$$

and if $p_j > 0$, then

$$\|(T_0-T)\,S_0^{p_j}\| \leqslant \alpha_0\,\|S_0\|^{p_j-1}.$$

Hence we have

$$\begin{aligned}
&\|S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\| \\
&\quad\leqslant \|S_0\|^{p_1}\,\|(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\| \\
&\quad\leqslant \|S_0\|^{q}\,\varepsilon_0^{q}\alpha_0^{q-1}\delta_{0,q}^{(k-1)/2-(q-1)}\,\|S_0\|^{\Sigma p_j-(k)} \\
&\quad= \|S_0\|^{q}\,\varepsilon_0^{q}\alpha_0^{q-1}\delta_{0,q}^{(k-2q+1)/2} \\
&\quad\leqslant \|P_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{q-2}\delta_{0,q}^{(k-2q+3)/2}
\end{aligned}$$

since $1 \leqslant \|P_0\|$ and $\|S_0\|\,\varepsilon_0\alpha_0 \leqslant \delta_{0,q}$.

Case 2: $p_1 = 0$. Among the $(k-1)/2$ pairs (p_j, p_{j+1}), $j = 2, 4, \ldots, k-1$, since exactly $(q-2)$ p_j's are zero, at least $(k-1)/2-(q-2) = (k-2q+3)/2$ pairs will have both p_j and p_{j+1} greater than zero. Since $\|S_0^{p_1}\| = \|P_0\|$, we obtain

$$\begin{aligned}&\|S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\|\\&\quad\leqslant \|P_0\|\,\|S_0\|^{q-1}\,\varepsilon_0^{q-1}\alpha_0^{q-2}\delta_{0,q}^{(k-2q+3)/2}.\end{aligned}$$

This proves (3.3).

In a similar manner, (3.4) can be proved. ∎

LEMMA 3.2. *If $v_0 < \frac{1}{4}$, then for any positive integer $i \geqslant 2$, we have*

$$\left\|\sum_{k=i}^{\infty}\sum_{\substack{(*,q),\,q\geqslant i,\\ p_{k+1}=0}} S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\right\| \leqslant M_i\,\|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\gamma_0,$$

where M_i is a constant depending on i only.

Proof. By using (3.1), (3.2) and the seies $\sum_{k=0}^{\infty} n_{k+1}x^k$ introduced in (2.1), it can be seen that

$$\begin{aligned}&\left\|\sum_{k=i}^{\infty}\sum_{\substack{(*,q),\,q\geqslant i,\\ p_{k+1}=0}} S_0^{p_1}(T_0-T)\,S_0^{p_2}\cdots(T_0-T)\,S_0^{p_{k+1}}\right\|\\&\quad\leqslant \|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\left[\sum_{k=i}^{4i-4} n_{k+1}\gamma_0^{k-i+1}+\gamma_0^{3i-3}\sum_{k=4i-3}^{\infty} n_{k+1}v_0^{k-4i+4}\right]\\&\quad\leqslant M_i\,\|P_0\|\,\|S_0\|^{i-1}\,\varepsilon_0^{i-1}\gamma_0\end{aligned}$$

since $v_0 < \frac{1}{4}$ and since the infinite power series is dominated by a series obtained by repeated differentiation of the series $\sum_{k=0}^{\infty} n_{k+1}v_0^k$. ∎

Now we prove the main result of this section which gives estimates for $\|PP_0 - P_0^j\|$, $j = 0, 1, 2, \ldots$.

PROPOSITION 3.3. *Let $p \geqslant 1$ be a fixed integer. Let $(\gamma_0)^{1/2}\,(u_0)^{1/4} < \frac{1}{4}$ and $(\delta_{0,p})^{1/2} < \frac{1}{4}$. Then for $m = 0, 1, \ldots, p-1$, we have*

$$\|PP_0 - P_0^{2m}\| = O(\|P_0\|\,\|S_0\|\,\varepsilon_0\max\{(u_0)^m, (\delta_{0,m+1})^m\}), \tag{3.13}$$

$$\|PP_0 - P_0^{2m+1}\| = O(\|P_0\|\,\|S_0\|\,\varepsilon_0\gamma_0\max\{(u_0)^m, (\delta_{0,m+1})^m\}). \tag{3.14}$$

Proof. Recall that

$$\begin{aligned} PP_0 - P_0 &= \sum_{k=1}^{\infty} P_0^{(k)} \\ &= -\sum_{k=1}^{\infty} \sum_{(*),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} \\ &= \sigma, \text{ say.} \end{aligned}$$

The sum σ can be decomposed into the sum σ_1 over the p_j's satisfying $(*, 1)$, plus the sum σ_2 over the p_j's satisfying $(*, 2)$, etc. Thus, for $q \geqslant 1$,

$$\sigma_q = -\sum_{k=q}^{\infty} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}.$$

We have

$$\begin{aligned} \sigma_1 &= -\sum_{k=1}^{\infty} \sum_{(*,1),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} \\ &= \sum_{k=1}^{\infty} S_0[(T_0 - T)\, S_0]^{k-1}\, (T_0 - T)\, P_0. \end{aligned}$$

The kth term of the above series is of the order of

$$\|S_0\|\, \beta_{0,1}^{(k-1)/2} \varepsilon_0. \tag{3.15}$$

Thus,

$$\|\sigma_1\| \leqslant \|S_0\|\, \varepsilon_0 \sum_{k=1}^{\infty} \beta_{0,1}^{(k-1)/2} \leqslant M_1\, \|S_0\|\, \varepsilon_0 = O(\|S_0\|\, \varepsilon_0)$$

since $\sqrt{\beta_{0,1}} \leqslant \sqrt{\delta_{0,p}} < \frac{1}{4} < 1$.

On putting $i = 2$ in Lemma 3.2, it follows that

$$\begin{aligned} \left\| \sum_{q=2}^{\infty} \sigma_q \right\| &= \left\| \sum_{q=2}^{\infty} \sum_{k=q}^{\infty} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} \right\| \\ &= \left\| \sum_{k=2}^{\infty} \sum_{\substack{(*,q),\, q \geqslant 2, \\ p_{k+1}=0}} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}} \right\| \\ &\leqslant M_2\, \|P_0\|\, \|S_0\|\, \varepsilon_0 \gamma_0, \end{aligned} \tag{3.16}$$

where M_2 is a constant. Thus,

$$\begin{aligned}\|\sigma\| &= \left\|\sum_{q=1}^{\infty} \sigma_q\right\| \\ &\leqslant \|\sigma_1\| + \left\|\sum_{q=2}^{\infty} \sigma_q\right\| \\ &\leqslant M_1 \|S_0\| \varepsilon_0 + M_2 \|P_0\| \|S_0\| \varepsilon_0 \gamma_0 \\ &\leqslant M \|P_0\| \|S_0\| \varepsilon_0. \end{aligned} \tag{3.17}$$

Therefore,

$$\|PP_0 - P_0\| = \|\sigma\| = O(\|P_0\| \|S_0\| \varepsilon_0).$$

This proves (3.13) for $m = 0$.

Next, for $j \geqslant 2$, we have

$$\begin{aligned}\|PP_0 - P_0^{j-1}\| \leqslant \Bigg\| \sum_{q=1}^{j-1} \Bigg[\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),\, p_{k+1}=0} & S_0^{p_1}(T_0 - T) \\ & \times S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}}\Bigg]\Bigg\| + \left\|\sum_{q=j}^{\infty} \sigma_q\right\|.\end{aligned}$$

As in (3.15), we see that for $j \geqslant 2$,

$$\begin{aligned}&\left\|\sigma_1 - \sum_{k=1}^{j-1} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}}\right\| \\ &\quad = \left\|\sigma_1 - \sum_{k=1}^{j-1} S_0[(T_0 - T) S_0]^{k-1} (T_0 - T) S_0^0\right\| \\ &\quad = \left\|\sum_{k=j}^{\infty} S_0[(T_0 - T) S_0]^{k-1} (T_0 - T) P_0\right\| \\ &\quad = O(\|S_0\| \varepsilon_0 (\beta_{0,1})^{(j-1)/2}). \end{aligned} \tag{3.18}$$

Fix q such that $2 \leqslant q \leqslant j-1$. We shall now give estimates for

$$\left\|\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}}\right\|.$$

Case $1 : j + 1 \leqslant 2q - 1$. We have

$$\left\|\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\right\|$$

$$= \left\|\sum_{k=j}^{\infty} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\right\|$$

$$\leqslant \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \left[\sum_{k=j}^{4q-4} n_{k+1} \gamma_0^{k-q+1} + \gamma_0^{3q-3} \sum_{k>4q-4} n_{k+1} v_0^{k-4q+4}\right],$$

(by (3.1) and (3.2) with $i = q$)

$$\leqslant M_{j,q} \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \gamma_0^{j-q+1}$$

$$\leqslant M_{j,q} \|P_0\|\, \|S_0\|\, \varepsilon_0 u_0^{(j-1)/2}, \tag{3.19}$$

where $M_{j,q}$ is a constant depending on j and q only.

Case 2: $j + 1 > 2q - 1$. Let j be odd. By (3.3) and (3.4),

$$\left\|\sum_{k=j}^{\infty} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T) \cdots (T_0 - T)\, S_0^{p_{k+1}}\right\|$$

$$\leqslant \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{q-2}$$

$$\times \left[n_{j+1}(\delta_{0,q})^{(j-2q+3)/2} + \alpha_0 \sum_{k=j+1} n_{k+1}(\delta_{0,q})^{(k-2q+2)/2}\right]$$

$$\leqslant M_{j,q} \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{q-2} (\delta_{0,q})^{(j-2q+3)/2}$$

(since $\sqrt{\delta_{0,q}} < \frac{1}{4}$)

$$\leqslant M_{j,q} \|P_0\|\, \|S_0\|\, \varepsilon_0 (\delta_{0,q})^{(j-1)/2}$$

(since $\|S_0\|\, \varepsilon_0 \alpha_0 \leqslant \delta_{0,q}$). (3.20)

If j is even, in a similar manner, we obtain

$$\left\|\sum_{k=j}^{\infty} \sum_{(*,q),\, p_{k+1}=0} S_0^{p_1}(T_0 - T)\, S_0^{p_2} \cdots (T_0 - T)\, S_0^{p_{k+1}}\right\|$$

$$\leqslant M_{j,q} \|P_0\|\, \|S_0\|^{q-1}\, \varepsilon_0^{q-1} \alpha_0^{q-1} (\delta_{0,q})^{(j-2q+2)/2}$$

$$\leqslant M_{j,q} \|P_0\|\, u_0 (\delta_{0,q})^{(j-2)/2}. \tag{3.21}$$

Thus, we see from (3.19)–(3.21) that for any q satisfying $2 \leqslant q \leqslant j - 1$, we have

$$\left\| \sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}} \right\|$$

$$\leqslant \begin{cases} M_{j,q} \|P_0\| \|S_0\| \varepsilon_0 u_0^{(j-1)/2}, \\ \qquad\qquad\qquad \text{if} \quad j+1 \leqslant 2q-1 \\ M_{j,q} \|P_0\| \|S_0\| \varepsilon_0 (\delta_{0,q})^{(j-1)/2}, \\ \qquad\qquad\qquad \text{if} \quad j+1 > 2q-1,\ (j+1) \text{ even} \\ M_{j,q} \|P_0\| u_0 (\delta_{0,q})^{(j-2)/2}, \\ \qquad\qquad\qquad \text{if} \quad j+1 > 2q-1,\ (j+1) \text{ odd.} \end{cases}$$

Let $m = 1,..., p-1$. If $j = 2m+1$ (so that $j+1$ is even), then

$$\begin{aligned} \|PP_0 - P_0^{2m}\| &= \|PP_0 - P_0^{j-1}\| \\ &\leqslant \left\| \sum_{q=1}^{j-1} \left[\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}} \right] \right\| \\ &\quad + \left\| \sum_{q=j}^{\infty} \sigma_q \right\| \\ &\leqslant \left\| \sigma_1 - \sum_{k=1}^{j-1} S_0[(T_0 - T) S_0]^{k-1} (T_0 - T) S_0^0 \right\| \\ &\quad + \left\| \sum_{q=2}^{m+1} \left[\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}} \right] \right\| \\ &\quad + \left\| \sum_{q=m+2}^{j-1} \left[\sigma_q - \sum_{k=q}^{j-1} \sum_{(*,q),p_{k+1}=0} S_0^{p_1}(T_0 - T) S_0^{p_2} \cdots (T_0 - T) S_0^{p_{k+1}} \right] \right\| \\ &\quad + \left\| \sum_{q=j}^{\infty} \sigma_q \right\|. \end{aligned}$$

By (3.18), (3.20), (3.19) and Lemma 3.2, we obtain

$$\begin{aligned} \|PP_0 - P_0^{2m}\| &= O(\|S_0\| \varepsilon_0 (\beta_{0,1})^m + \|P_0\| \|S_0\| \varepsilon_0 (\delta_{0,m+1})^m \\ &\quad + \|P_0\| \|S_0\| \varepsilon_0 (u_0)^m + \|P_0\| \|S_0\| \varepsilon_0 (u_0)^m) \\ &= O(\|P_0\| \|S_0\| \varepsilon_0 \max\{(u_0)^m, (\delta_{0,m+1})^m\}). \end{aligned}$$

This proves (3.13).

Similarly, it can be proved that for $m = 1,..., p-1$,

$$\|PP_0 - P_0^{2m+1}\| = O(\|P_0\| \|S_0\| \varepsilon_0 \gamma_0 \max\{(u_0)^m, (\delta_{0,m+1})^m\}),$$

by using (3.18), (3.21), (3.19) and Lemma 3.2, and noting that $u_0 = \|S_0\| \varepsilon_0 \gamma_0$. ∎

Remark 3.4. In case $\gamma_0 = \max\{\|S_0\|\,\varepsilon_0, \alpha_0\}$ is small, we see from (3.13) and (3.14) that the $(2m+1)$th approximation P_0^{2m+1} of PP_0 is better than the $2m$th approximation P_0^{2m}, $m = 0, 1,..., p-1$. However, if γ_0 is not small but u_0 and $\delta_{0,p}$ are small, then the approximations P_0^{2m} and P_0^{2m+1} are of the same order, while the $(2m+2)$th approximation $(m = 0,..., p-2)$ is better than these two because $\delta_{0,m+1} \leqslant \delta_{0,m+2}$.

4. Norm and Collectively Compact Convergences

Throughout this section, we assume that λ is a simple isolated eigenvalue of T.

THEOREM 4.1. *Let $\|T_n - T\| \to 0$ as $n \to \infty$. There exists n_0 such that for every fixed $n \geqslant n_0$, P_n^j converges to PP_n as $j \to \infty$ in a geometric fashion. In fact,*

$$\|PP_n - P_n^j\| = O(\gamma_n^{j+1}), \qquad j \geqslant 0. \tag{4.1}$$

Proof. $\|T_n - T\| \to 0$ implies that $\gamma_n = \max\{\|S_n\|\,\varepsilon_n, \alpha_n\} \to 0$. Since

$$\beta_{n,k} = \frac{\|(T_n - T)\, S_n^k (T_n - T)\, S_n\|}{\|S_n\|^{k-1}} \leqslant \|(T_n - T)\, S_n\|^2 \leqslant \gamma_n^2,$$

we see that

$$\delta_{n,k} = \max\{\|S_n\|\,\varepsilon_n \alpha_n, \beta_{n,1},..., \beta_{n,k}\} \leqslant \gamma_n^2$$

for all k. Choose n_0 such that $\gamma_n < \frac{1}{4}$ for all $n \geqslant n_0$. Then for $n \geqslant n_0$, $(\gamma_n)^{1/2}\,(u_n)^{1/4} \leqslant \gamma_n < \frac{1}{4}$ and $(\delta_{n,k})^{1/2} \leqslant \gamma_n < \frac{1}{4}$ for all k. Thus, the assumptions in Proposition (3.3) are satisfied for all $p \geqslant 1$, if T_0 is replaced by T_n with $n \geqslant n_0$. Then (4.1) follows from (3.13) and (3.14). ∎

THEOREM 4.2. *Let $T_n \to^{\text{c.c.}} T$. Let $p \geqslant 1$ be a fixed integer. There exists n_0 such that for every fixed $n \geqslant n_0$, and for $m = 0,..., p-1$, we have*

$$\begin{aligned} \|PP_n - P_n^{2m}\| &= O(\|P_n\|\,\|S_n\|\,\varepsilon_n \max\{(u_n)^m, (\delta_{n,m+1})^m\}), \\ \|PP_n - P_n^{2m+1}\| &= O(\|P_n\|\,\|S_n\|\,\varepsilon_n \gamma_n \max\{(u_n)^m, (\delta_{n,m+1})^m\}). \end{aligned} \tag{4.2}$$

Thus, P_n^j approximates PP_n in a semigeometric fashion for $j = 0,..., 2p-1$.

Proof. It follows from Proposition 2.1 that if $T_n \to^{\text{c.c.}} T$, then $\delta_{n,p} \to 0$. Also, $T_n \to^{\text{c.c.}} T$ implies that γ_n is bounded and $u_n = \|S_n\|\,\varepsilon_n \gamma_n \to 0$. Choose n_0 such that for all $n \geqslant n_0$, we have $(\gamma_n)^{1/2}\,(u_n)^{1/4} < \frac{1}{4}$ and $(\delta_{n,p})^{1/2} < \frac{1}{4}$. The proof of the theorem then follows from Proposition 3.3 with T_0 replaced by T_n, $n \geqslant n_0$. ∎

Theorems 4.1 and 4.2 show that if T_n converges to T in the norm, then every approximation P_n^j of PP_n improves upon the earlier approximation P_n^{j-1}, whereas if T_n converges to T in the collectively compact fashion, then this improvement is obtained only in steps of 2. The proofs of these theorems depend only on the facts that under norm convergence $\gamma_n \to 0$, while under collectively compact convergence $u_n \to 0$ and $\delta_{n,p} \to 0$ for any fixed p. Hence we obtain the following general results.

THEOREM 4.3. *Let T be a closed operator with domain D and T_n a sequence of closed operators whose domains contain D and $T_n x \to Tx$ for all $x \in D$. Let λ be a simple isolated eigenvalue of T.*

(a) *If*

$$\|(T_n - T) R(z)\| \to 0 \tag{4.3}$$

for $z \in \rho(T)$, then for a fixed large enough n, the sequence P_n^j converges (as $j \to \infty$) to PP_n in a geometric manner.

(b) *If*

$$\|(T_n - T) R(z)^k (T_n - T) R(z)\| \to 0 \tag{4.4}$$

for $z \in \rho(T)$, $1 \leqslant k \leqslant p$, then for a fixed large enough n, P_n^j approximates PP_n in a semigeometric manner for $j = 0,..., 2p-1$.

Proof. (a) If $\|(T_n - T) R(z)\| \to 0$, then $P_n \to^{\text{c.c.}} P$ (Chetelin [2, Proposition 5.23]), so that $\varepsilon_n = \|(T_n - T) P_n\| \to 0$. From the second Neumann series of $R_n(z)$ it follows that $\|(T_n - T) R_n(z)\| \to 0$, $z \in \Gamma$. Hence

$$\alpha_n = \|(T_n - T) S_n\| \leqslant \max_{z \in \Gamma} \|(T_n - T) R_n(z)(I - P_n)\| \to 0.$$

Thus, $\gamma_n = \max\{\|S_n\| \varepsilon_n, \alpha_n\} \to 0$. Now, the result follows as in the proof of Theorem 4.1.

(b) It can be seen from the proof of Proposition 2.1 that if (4.4) holds, then $\delta_{n,p} \to 0$ as $n \to \infty$. Now

$$\|(T_n - T) R(z)(T_n - T) R(z)\| \to 0$$

implies that $\varepsilon_n = \|(T_n - T) P_n\| \to 0$ (Chatelin [3, Lemma 5]) and $\alpha_n = \|(T_n - T) S_n\|$ is bounded. Thus, $u_n \to 0$. The desired result then follows as in the proof of Theorem 4.2. ∎

Remark 4.4. Condition (4.3) in Theorem 4.3(a) is satisfied in the case of neighbouring approximation and in the case of approximation by a holomorphic family of type (A) (Chatelin and Lemordant [3, Theorem 2bis]).

Remark 4.5. Condition (4.4) in Theorem 4.3(b) corresponding to the case $p = 1$, i.e.,

$$\|[(T_n - T)R(z)]^2\| \to 0$$

was considered in [3] under the name strong convergence, and estimates similar to

$$\|PP_n - P_n\| = O(\varepsilon_n),$$
$$\|PP_n - P_n - S_n(T_n - T)P_n\| = O(\alpha_n \varepsilon_n)$$

were obtained (Chatelin and Lemordant [3, (5.3) and (5.4)]). In [6], the next (i.e., second) approximation of PP_n was considered and an error bound similar to

$$\begin{aligned}\|PP_n - P_n - S_n(T_n - T)P_n - S_n(T_n - T)S_n(T_n - T)P_n \\ + S_n^2(T_n - T)P_n(T_n - T)P_n + P_n(T_n - T)S_n^2(T_n - T)P_n\| \\ = O(\alpha_n \varepsilon_n^2, \beta_{n,1}\varepsilon_n, \beta_{n,2}\varepsilon_n, \varepsilon_n^3)\end{aligned}$$

was obtained (Kulkarni and Limaye [6, (4.7)]).

Also, note that if $T_n R(z) \to^{\text{c.c.}} TR(z)$ for $z \in \rho(T)$, then condition (4.4) in Theorem 4.4(b) is satisfied for all p (just as in the case of $T_n \to^{\text{c.c.}} T$).

5. Approximation of Eigenvectors, Numerical Experiments

Let $T_n \to^{\text{c.c.}} T$, and ϕ_n be an eigenvector of T_n associated with λ_n. Then

$$P\phi_n = PP_n\phi_n = P_n\phi_n + \sum_{k=1}^{\infty} P_n^{(k)}\phi_n = \phi_n + \sum_{k=1}^{\infty} P_n^{(k)}\phi_n$$

is an eigenvector of T associated with λ whenever it is not zero.

It follows from Theorem 4.2 that under the assumption of the collectively compact convergence, $P_n^{2m}\phi_n$ and $P_n^{2m+1}\phi_n$ are approximations of $P\phi_n$ of the same order, whereas $P_n^{2m+2}\phi_n$ improves upon both of them.

Now we consider another eigenvector of T corresponding to λ, given by the Rayleigh–Schrodinger series

$$\lambda = \sum_{k=0}^{\infty} \lambda_n^{(k)} \qquad \text{and} \qquad \phi = \sum_{k=0}^{\infty} \phi_n^{(k)},$$

where $\lambda_n^{(0)} = \lambda_n$, $\phi_n^{(0)} = \phi_n$, and for $k = 1, 2, \ldots$,

$$\lambda_n^{(k)} \phi_n = P_n(T - T_n)\, \phi_n^{(k-1)},$$

$$\phi_n^{(k)} = S_n \left[-(T - T_n)\, \phi_n^{(k-1)} + \sum_{i=1}^{k} \lambda_n^{(i)} \phi_n^{(k-i)} \right].$$

Consider the jth approximation of ϕ:

$$\phi_n^j = \sum_{k=0}^{j} \phi_n^{(k)}, \qquad j = 0, 1, 2, \ldots.$$

It can be proved that ϕ_n^j's approximate ϕ is a semigeometric fashion [7, 8].

For two integral operators we compute the zeroth, the first, and the second approximations of $P\phi_n$ and ϕ. The numerical results show a shift in the successive approximations of $P\phi_n$ as against those of ϕ.

EXAMPLE 5.1. $X = C([0, 1])$. T is an integral operator with kernel

$$\begin{array}{ll} s(1-t), & \text{if} \quad s \leqslant t, \\ t(1-s), & \text{if} \quad t < s. \end{array}$$

T is approximated by Nystrom's method T_n^N [4]. The approximate

TABLE I

$n = 20$

(a)

i	$\lVert P\phi_n - P_n^0 \phi_n \rVert$	$\lVert P\phi_n - P_n^1 \phi_n \rVert$	$\lVert P\phi_n - P_n^2 \phi_n \rVert$
1	5.5×10^{-4}	7.2×10^{-4}	1.2×10^{-5}
2	2.2×10^{-3}	2.8×10^{-3}	1.8×10^{-4}
3	5.1×10^{-3}	6.3×10^{-3}	8.6×10^{-4}
4	9.2×10^{-3}	1.1×10^{-2}	2.5×10^{-3}

(b)

i	$\lVert \phi - \phi_n^0 \rVert$	$\lVert \phi - \phi_n^1 \rVert$	$\lVert \phi - \phi_n^2 \rVert$
1	1.5×10^{-3}	9.4×10^{-6}	3.5×10^{-6}
2	6.3×10^{-3}	1.5×10^{-4}	—
3	1.5×10^{-2}	7.9×10^{-4}	2.3×10^{-4}
4	2.6×10^{-2}	2.5×10^{-3}	—

TABLE II

$i = 1$

(a)

n	$\|P\phi_n - P_n^0\phi_n\|$	$\|P\phi_n - P_n^1\phi_n\|$	$\|P\phi_n - P_n^2\phi_n\|$
4	3.4×10^{-3}	3.4×10^{-3}	2.7×10^{-5}
8	6.1×10^{-4}	6.0×10^{-4}	9.8×10^{-7}
12	2.4×10^{-4}	2.4×10^{-4}	1.6×10^{-7}

(b)

n	$\|\phi - \phi_n^0\|$	$\|\phi - \phi_n^1\|$	$\|\phi - \phi_n^2\|$
4	4.9×10^{-3}	2.4×10^{-5}	3.3×10^{-5}
8	9.5×10^{-4}	2.0×10^{-5}	1.7×10^{-5}
12	3.9×10^{-4}	1.8×10^{-5}	1.7×10^{-5}

quadrature formula is the Gauss formula with two points. $T_n^N \to^{\text{c.c.}} T$ (Chatelin [4, Theorem 2]). T is replaced by T_{100}^N. i is the index of the eigenvalue, i.e., the case $i = 1$ corresponds to the largest eigenvalue, the case $i = 2$ corresponds to the second largest eigenvalue, etc. Computations in Table I(a) are done by us in double precision on DEC-System 10. Computations in Table I(b) are taken for the sake of comparison from Chatelin [4, Table 4] and Redont [8, Table 12].

EXAMPLE 5.2. $X = C([0, 1])$. T is an integral operator with kernel

$$e^{st}, \qquad 0 \leqslant s, t \leqslant 1.$$

Divide $[0, 1]$ into $n - 1$ subintervals by $t_j = i/(n-1), j = 0, 1, \ldots, n-1$. Let π_n be the linear interpolatory projection. T is approximated by Galerkin's method, $T_n^G = \pi_n T \pi_n$. $T_n^G \to^{\text{c.c.}} T$ (Chatelin [4, Theorem 1]). We replace T by T_{60}^G. Computations are performed in double precision on DEC-System 10 and are tabulated in Tables II.

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